

Quantum Diffraction of Neutrino

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Abstract

The probability to detect a neutrino produced in pion decay at a finite distance exhibits unique interference properties that depend on the absolute mass of the neutrino. We describe the neutrino, lepton, and pion by a many-body wave function and find the probability is subject to a large finite-size correction. Its rate at a distance L is expressed as $\Gamma_0 + \tilde{g}(\omega_\nu L/c)\Gamma_1$, where $\tilde{g}(\omega_\nu L/c)$ is the universal function, $\omega_\nu = m_\nu^2 c^4 / (2E_\nu \hbar)$, c is the speed of light, and Γ_0 is a constant computed with the standard plane-wave S-matrix. The finite-size correction is rigorously computed using wave packets via the light-cone singularity of a system composed of the pion and charged lepton and reveals the diffraction pattern of a single quantum. We discuss the implications of this correction for the muon-neutrino and electron-neutrino reactions. With sufficient statistics, the neutrino diffraction would supply the absolute mass of the neutrino.

1 Neutrino interference

Interference phenomena involving photons, electrons, neutrons, and other heavy elements are useful for testing quantum mechanics and other basic principles. Neutrinos are elementary particles that are produced in weak decays of pions and other weak processes and are described by quantum waves that follow the superposition principle. Since neutrinos interact extremely weakly with matter, the probability to detect neutrinos at a finite distance exhibits a single quantum interference that makes the probability receive an unusual finite-size correction. This pattern could, via its unique behavior, provide an absolute value of the neutrino mass, which is at present an unknown physical quantity.

A system composed of a pion, charged lepton, and neutrino are described by the following Lagrangian of a field of pion, $\varphi(x)$, of charged lepton, $l(x)$, and of neutrino, $\nu(x)$,

$$\begin{aligned} L &= L_0 + L_{int}, \\ L_0 &= \partial_\mu \varphi^* \partial^\mu \varphi - m_\pi^2 \varphi^* \varphi + \bar{l}(\gamma \cdot p - m_l)l + \bar{\nu}(\gamma \cdot p - m_\nu)\nu, \\ L_{int} &= g J_{hadron}^{V-A} \times J_{lepton}^{V-A}, \quad g = G_F/\sqrt{2}, \end{aligned} \quad (1)$$

where G_F is the Fermi coupling constant and J_i^{V-A} is $V - A$ current. The system is invariant under a Lorentz transformation and momenta can be any value from $-\infty$ to $+\infty$. A pion and decay products are expressed by a many body wave function $|\Psi\rangle$ that follows a Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = (H_0 + H_{int}) |\Psi\rangle, \quad (2)$$

where the free part, H_0 , and the interaction part, H_{int} , are derived from the previous Lagrangian. A time-dependent solution in the first order of H_{int} is

$$\begin{aligned} |\Psi\rangle &= e^{-i\frac{E_0}{\hbar}t} \left(|\psi^{(0)}\rangle + \int d\beta D(\omega, t) |\beta\rangle \langle\beta| H_{int} |\psi^{(0)}\rangle \right), \\ \omega &= E_\beta - E_0, \quad H_0 |\beta\rangle = E_\beta |\beta\rangle, \quad H_0 |\psi^{(0)}\rangle = E_0 |\psi^{(0)}\rangle, \end{aligned} \quad (3)$$

where $d\beta$ is a measure for a complete set of $|\beta\rangle$, and

$$D(\omega, t) = \frac{e^{-i\frac{\omega}{\hbar}t} - 1}{\omega}. \quad (4)$$

In Eq. (3), all the possible states of $|\beta\rangle$ that couple with H_{int} must be included.

$D(\omega, t)$ at $t = \infty$ and an average over a finite interval, Δt of $\omega\Delta t \gg 1$, behave as

$$D(\omega, t) = -2\pi i \delta(\omega); \quad t \rightarrow \infty, \quad (5)$$

$$\text{average } D(\omega, t) = -\frac{1}{\omega}; \quad \text{finite } t. \quad (6)$$

Thus the wave function at $t = \infty$ and the average over a finite interval at a finite t are

$$|\Psi_\infty\rangle = e^{-i\frac{E_0}{\hbar}t} \left(|\psi^{(0)}\rangle - 2\pi i \int d\beta \delta(E_\beta - E_0) |\beta\rangle \langle\beta| H_{int} |\psi^{(0)}\rangle \right), \quad (7)$$

$$|\Psi_{average}\rangle = e^{-i\frac{E_0}{\hbar}t} \left(|\psi^{(0)}\rangle - \int d\beta \frac{1}{\omega} |\beta\rangle \langle\beta| H_{int} |\psi^{(0)}\rangle \right), \quad (8)$$

and satisfy in the first order of H_{int} ,

$$H|\Psi_\infty\rangle = E_0|\Psi_\infty\rangle, \quad H_0|\Psi_\infty\rangle = E_0|\Psi_\infty\rangle, \quad (9)$$

$$H|\Psi_{average}\rangle = E_0|\Psi_{average}\rangle, \quad H_0|\Psi_{average}\rangle \neq E_0|\Psi_{average}\rangle. \quad (10)$$

In both cases, the frequency is E_0/\hbar and the total energy defined by H is E_0 . The total energy is conserved. The state of a momentum eigenstate at $t = 0$

$$|\psi^{(0)}\rangle = |\vec{p}_\pi\rangle, \quad (11)$$

evolves according to Eq. (3) and is a coherent superposition of $|\psi^{(0)}\rangle$ and plane waves of a lepton and neutrino with the particular weight. The state at a finite t is different from that at $t = \infty$, which is used for an ordinary calculation of the decay rate.

At the infinite time t , the state becomes a superposition of $|\psi^{(0)}\rangle$ and $|\beta_0\rangle$ of the same kinetic energy $E_{\beta_0} = E_0$. Since the state $|\beta_0\rangle$ has the same kinetic energy, this is regarded as a state composed of free particles. The decay products follow dynamics of classical particles, and their norm shows the probability, which increases linearly with t , hence a pion decays with a constant rate like a classical particle.

At a finite time t , on the other hand, the state is a superposition of $|\psi^{(0)}\rangle$ and those $|\beta\rangle$ of an arbitrary value of $E_\beta \geq 0$. Since the kinetic energy E_β is different from E_0 and not a constant, the states in this region are different

from the state composed of free particles. The particle picture does not hold, and the wave nature remains here.

The evolution of the state Eq. (3) in time is similar to an evolution of a stationary state of a system with potential or obstacle in \vec{x} , where the kinetic energy varies due to a potential energy. A superposition of plane waves is formed and reveals interference of waves of many components, which we call diffraction in this paper. A superposition of waves Eq. (3) at a finite t could show a similar phenomenon. In fact, the kinetic energy of waves composed of a charged lepton and neutrino in Eq. (3) at a finite t is not constant. Hence this could show a non-uniform space-time dependent behavior, which is a feature of diffraction. Now the weight of states is determined by H_{int} and this diffraction phenomenon should have a universal property. After passing this period, the state has the energy E_0 and becomes to be particle-like again at $t \rightarrow \infty$, in the same way as the stationary wave at $|\vec{x}| \rightarrow \infty$.

Physical quantities are measured by observing the lepton or neutrino. Probability to detect the particle is finite experimentally and a theoretical value must be finite and unique with an amplitude that satisfies boundary conditions of experiments. The neutrino is identified from its reactions with nucleus that are caused by the weak interaction and its properties are determined by the probabilities that these processes occur. Hence the final state of the transition processes is expressed by a wave packet of the nuclear size. Accordingly the probability to detect this neutrino is expressed by the wave packets. Wave packets that satisfy free wave equations and are localized in space are important for rigorously defining scattering amplitude [1, 2].

A standard method to calculate the decay rate that uses the transition amplitudes of plane waves with asymptotic boundary conditions at time $t = \pm\infty$ gives the asymptotic quantities at $T = \infty$, where T is a time interval between in and out states. But that gives neither quantities at a finite T nor a position-dependent probability. This is because the boundary conditions are different and the plane waves are translationally invariant. To compute the position-dependent probability, the wave function expressing entire pion-decay process, which begins as a superposition of a one-pion state and the state of the decay products, and satisfying the boundary conditions are necessary. The amplitude of this situation is expressed by $S[T]$ satisfying the boundary conditions at a finite T given below.

$S[T]$ is defined by Møller operators at a finite T , $\Omega_{\pm}(T)$, as $S[T] = \Omega_{-}^{\dagger}(T)\Omega_{+}(T)$. $\Omega_{\pm}(T)$ are expressed by a free Hamiltonian H_0 and a total

Hamiltonian H by $\Omega_{\pm}(T) = \lim_{t \rightarrow \mp T/2} e^{iHt} e^{-iH_0 t}$. From this expression, $S[T]$ satisfies

$$[S[T], H_0] = i \left\{ \frac{\partial}{\partial T} \Omega_{-}^{\dagger}(T) \right\} \Omega_{+}(T) - i \Omega_{-}^{\dagger}(T) \frac{\partial}{\partial T} \Omega_{+}(T). \quad (12)$$

Thus a kinetic energy defined by H_0 is not conserved except at $T = \infty$ and a matrix element of $S[T]$ between two eigenstates of H_0 , $|\alpha\rangle$ and $|\beta\rangle$ of eigenvalues E_{α} and E_{β} , is decomposed into two components

$$\langle \beta | S[T] | \alpha \rangle = \langle \beta | S^{(n)}[T] | \alpha \rangle + \langle \beta | S^{(d)}[T] | \alpha \rangle, \quad (13)$$

where $S^{(n)}$ satisfies $E_{\beta} = E_{\alpha}$ and $S^{(d)}$ satisfies $E_{\beta} \neq E_{\alpha}$. At $T \rightarrow \infty$, the right-hand side of Eq. (12) vanishes and so does the second term of Eq. (13). Hence the energy defined by H_0 is conserved. At a finite T , on the other hand, the right-hand side of Eq. (12) and second term of Eq. (13) do not vanish. The states of $E_{\beta} \neq E_{\alpha}$ couple and give the finite-size correction. The states of the different eigenvalues of H_0 are orthogonal each other and the cross term of the first and second terms of Eq. (13) in a square of the modulus vanishes. Consequently the finite-size correction to the probability becomes positive semi-definite. Unitarity $S[T]S^{\dagger}[T] = 1$ is satisfied and ensures the conservation of probability.

Innumerable states in the ultra-violet energy region in Eq. (3) propagate with the speed of light and couple to $S[T]$, hence they necessarily give a finite-size correction to a particle with small mass. Although the finite-time-interval effect has been considered insignificant and ignored heretofore, the correction is actually non-negligible and important especially to recent and future neutrino experiments. We show herein that the probability to detect a neutrino is studied with $S[T]$ and is affected by the large finite-size correction in a macroscopic area. The diffraction of a neutrino is a main part of the finite-size correction for a system of a pion and charged lepton, and neutrino wave, and becomes visible in macroscopic scales.

Because an asymptotic boundary condition in scattering or decay processes [1, 2] is satisfied manifestly with wave packets, textbooks on scattering theory [3, 4, 5] emphasize the importance of wave packets. They give furthermore proofs that the asymptotic values are computed with plane waves. For computing non-asymptotic value that depends on a time interval, the wave packets are necessary. In the process of detecting neutrino, the neutrino wave packet [9, 10, 11] is well localized and expresses a nucleon wave function in

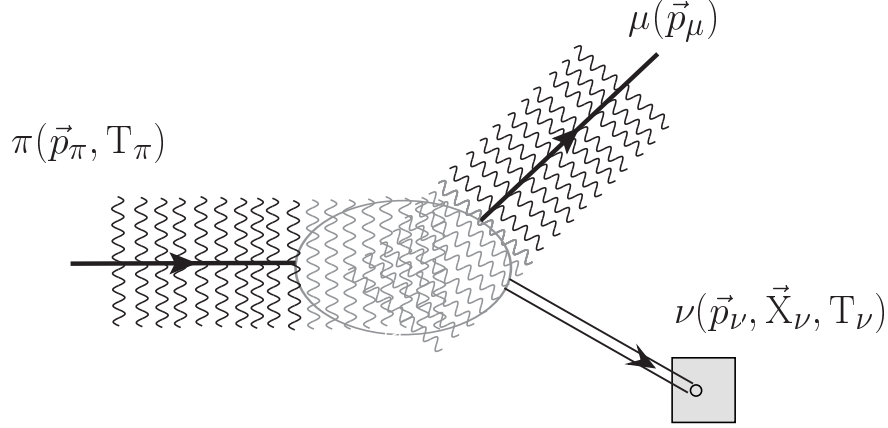


Figure 1: A pion of a momentum \vec{p}_π decays to a muon of a momentum \vec{p}_μ and a neutrino of a wave packets $(\vec{p}_\nu, \vec{X}_\nu)$. Since the neutrino wave packet becomes finite along a narrow region of a velocity \vec{v}_ν , a coordinate \vec{x} is integrated over this region. This quantum mechanical amplitude resembles diffraction of a classical wave that passes through a hole.

the nucleus with which the neutrino interacts [13, 14, 15, 16, 17, 18, 19, 20]. From the probability obtained with this amplitude, the physical quantities measured in experiments are calculated, such as the number of events, the flux, and others. The initial pion is expressed by a large wave packet, so the plane-wave approximation is very good and is therefore used in this paper. The unobserved muon is also expressed by a plane wave.

Mass-squared differences δm_ν^2 are negligibly small [6], and their central value is currently unknown, so we study the situation in which the average-mass-squared \bar{m}_ν^2 satisfies, $\bar{m}_\nu^2 \gg \delta m_\nu^2$. To begin, we present the one-flavor case. Extension to the general case is straightforward.

2 Position-dependent probability

We now compute the probabilities to detect neutrinos and charged leptons, l , at a finite distance. An amplitude to detect a neutrino of an average momentum \vec{p}_ν at \vec{X}_ν in the decay of a pion prepared at $t = T_\pi$ of a momentum \vec{p}_π , Fig. (1), is expressed as $T = \int d^4x \langle l, \nu | H_w(x) | \pi \rangle$, where a lepton l has a momentum \vec{p}_l and a neutrino is expressed by a wave packet. These states are

expressed as $|\pi\rangle = |\vec{p}_\pi, T_\pi\rangle$, $|l, \nu\rangle = |\mu, \vec{p}_l; \nu, \vec{p}_\nu, \vec{X}_\nu, T_\nu\rangle$. T is written with the hadronic matrix element of $V - A$ current and Dirac spinors

$$T = \int d^4x d\vec{k}_\nu N_1 \langle 0 | J_{V-A}^\mu(0) | \pi \rangle \bar{u}(\vec{p}_l) \gamma_\mu (1 - \gamma_5) \nu(\vec{k}_\nu) \\ \times \exp \left[-i(p_\pi - p_l) \cdot x / \hbar + i k_\nu \cdot (x - X_\nu) / \hbar - \frac{\sigma_\nu}{2} (\vec{k}_\nu - \vec{p}_\nu)^2 \right], \quad (14)$$

where $N_1 = ig(\sigma_\nu/\pi)^{\frac{4}{3}}(m_l m_\nu/E_l E_\nu)^{\frac{1}{2}}$. The time t is integrated over the region $T_\pi \leq t \leq T_\nu$. σ_ν is the size of the neutrino wave packet and is estimated from the size of a nucleus. For the sake of simplicity, we use the Gaussian form of the wave packet in this paper. The result for the finite-size correction is the same in general wave packets, as will be verified in Appendix C. The amplitude T depends on the momenta of the final state and also on coordinates (T_ν, \vec{X}_ν) and T_π .

Integrating over \vec{k}_ν in Eq. (14), we obtain a Gaussian function of \vec{x} about the moving center with the velocity $\vec{v}_\nu = \vec{p}_\nu c^2/E_\nu$, $\vec{x}_0 = \vec{X}_\nu + \vec{v}_\nu(t - T_\nu)$, which vanishes at large $|\vec{x} - \vec{x}_0|$ and satisfies the asymptotic boundary condition. The integrand becomes finite along a narrow space-time region of the velocity \vec{v}_ν , and is similar to classical waves that pass through a small hole, which show diffraction due to boundary conditions. If we integrate over the space coordinate \vec{x} and the time t in the finite-time interval $T = T_\nu - T_\pi$, we find the amplitude proportional to

$$\frac{\sin\left(\frac{\omega T}{2\hbar}\right)}{\omega} \exp\left(-\frac{\sigma_\nu}{2} \delta \vec{p}^2\right), \quad \omega = \delta E - \vec{v}_\nu \cdot \delta \vec{p}, \\ \delta E = E_l(\vec{p}_\mu) + E_\nu(\vec{p}_\nu) - E_\pi(\vec{p}_\pi), \quad \delta \vec{p} = \vec{p}_\nu + \vec{p}_l - \vec{p}_\pi. \quad (15)$$

In the limit σ_ν and $T \rightarrow \infty$, the amplitude is proportional to the four dimensional delta function $(2\pi)^4 \delta^{(4)}(\delta p)$, and becomes non-zero in the finite kinematical region of the final states. If σ_ν is finite, $\delta \vec{p}$ is not strictly zero and ω deviates from δE . At a large T , the amplitude gets dominant contributions from $\omega \approx 0$. Since the frequency ω is $\delta E - \vec{v}_\nu \cdot \delta \vec{p}$, which is equivalent to that of a system moving with velocity \vec{v}_ν , $\omega = 0$ has a solution of $\delta \vec{p} \neq 0$ and $\delta E \neq 0$ (see Appendix A) in addition to the normal solution, $\delta \vec{p} = 0$ and $\delta E = 0$. Thus the number of states at $\omega \approx 0$ becomes much larger than that of $\delta E \approx 0$ and the probability integrated over the final states approaches the asymptotic value very slowly with T . Hence the quantum mechanical amplitude to detect a neutrino at a finite T has a large finite-size correction.

The amplitude in quantum mechanics that includes the detection process, Eq. (14), is in fact almost equivalent to the classical waves of the diffraction caused by the boundary conditions. Accordingly it would be reasonable to call the present phenomenon “neutrino diffraction”. Being a purely quantum mechanical phenomenon, neutrino diffraction has various unique properties that are different from the diffraction of the classical waves.

The total probability is an integral of a square of the modulus of the amplitude over the complete set of final states [9]

$$P = \int d\vec{X}_\nu \frac{d\vec{p}_\nu}{(2\pi)^3} \frac{d\vec{p}_l}{(2\pi)^3} \sum_{s_1, s_2} |T|^2, \quad (16)$$

where the momenta of the neutrino and muon are integrated over whole positive energy region and the position of the wave packet is integrated over the region of the detector. This depends on a time interval T . Hereafter the natural unit, $c = \hbar = 1$, is taken in majority of places, but c and \hbar are written explicitly when it is necessary.

To rigorously compute the finite-size correction in a consistent manner with the Lorentz invariance, we express the probability Eq. (16) with a correlation function $\Delta_{\pi,l}(x_1, x_2)$ of the coordinates. The total probability is finite and the order of integrations of the coordinates and momentum is interchangeable. Here the muon momentum is integrated first for fixed x_i , after summing over the spin, the probability is written in the form

$$P = \int d\vec{X}_\nu \frac{d\vec{p}_\nu}{(2\pi)^3} \frac{N_2}{E_\nu} \int d^4x_1 d^4x_2 \exp \left[-\frac{1}{2\sigma_\nu} \sum_i (\vec{x}_i - \vec{x}_i^0)^2 + i\phi(\delta x) \right] \Delta_{\pi,l}(\delta x),$$

$$\vec{x}_i^0 = \vec{X}_\nu + \vec{v}_\nu(t_i - T_\nu), \quad \delta x = x_1 - x_2, \quad \phi(\delta x) = p_\nu \cdot \delta x, \quad (17)$$

where $N_2 = g^2 (4\pi/\sigma_\nu)^{\frac{3}{2}} V^{-1}$, V is a normalization volume for the initial pion, and

$$\Delta_{\pi,l}(\delta x) = \frac{1}{(2\pi)^3} \int \frac{d\vec{p}_l}{E(\vec{p}_l)} \{ 2(p_\pi \cdot p_\nu)(p_\pi \cdot p_l) - m_\pi^2(p_l \cdot p_\nu) \} e^{-i(p_\pi - p_l) \cdot \delta x}. \quad (18)$$

It is important to notice that the kinematical regions for the momenta of the final state became finite for the plane waves at $T \rightarrow \infty$ due to the delta function which resulted from the x integration. The muon momentum of this

expression is that of the asymptotic state and necessarily a part of those that satisfy the energy and momentum conservation. Now the integrations of the coordinates and momenta were interchanged in the amplitude of a finite T, in which the states of non-conserving energy of H_0 couple. In Eq. (18) the coordinates are fixed and the muon momentum is not subject to the energy-momentum conservation. Hence the momentum is integrated over the entire positive-energy region as in Eq. (16). The finite-size correction is necessarily generated and computed from those states that do not satisfy the energy momentum conservation.

$\Delta_{\pi,l}(\delta x)$ is expressed by the four-dimensional integrals over the variable $q = p_l - p_\pi$, as the sum of the light-cone singularity, $\delta(\delta x^2)$, and less singular and regular terms that are described by Bessel functions (see Appendix B). Thus the correlation function, $\Delta_{\pi,l}(\delta x)$, is expressed as

$$\Delta_{\pi,l}(\delta x) = 2i \left\{ m_\pi^2 p_\nu \cdot \left(p_\pi + i \frac{\partial}{\partial \delta x} \right) - 2i(p_\pi \cdot p_\nu) \left(p_\pi \cdot \frac{\partial}{\partial \delta x} \right) \right\} \\ \times \left[\frac{\epsilon(\delta t)}{4\pi} \delta(\lambda) + I_1^{regular} + I_2 \right], \quad \lambda = (\delta x)^2 = \delta t^2 - \delta \vec{x}^2, \quad (19)$$

where $I_1^{regular}$ is composed of Bessel functions and has a singularity of the form $1/\lambda$ around $\lambda = 0$ and decrease as $e^{-\tilde{m}\sqrt{|\lambda|}}$ or oscillates as $e^{i\tilde{m}\sqrt{|\lambda|}}$ at large $|\lambda|$ (see Appendix B), where $\tilde{m}^2 = m_\pi^2 - m_l^2$. $\epsilon(\delta t)$ is a sign function and $\delta(\lambda)$ is Dirac's delta function.

Next, we substitute Eq. (19) into Eq. (17) and integrate over \vec{x}_1 and \vec{x}_2 with the center coordinate $X^\mu = (x_1^\mu + x_2^\mu)/2$ and the relative coordinate $\vec{r} = \vec{x}_1 - \vec{x}_2$. We have the expression from the light-cone singular term

$$J_{\delta(\lambda)} = C_{\delta(\lambda)} \frac{\epsilon(\delta t)}{|\delta t|} \exp \left[i \bar{\phi}_c(\delta t) - \frac{m_\nu^4 c^8}{16 \sigma_\nu E_\nu^4} \delta t^2 \right], \\ C_{\delta(\lambda)} = \frac{(\sigma_\nu \pi)^{\frac{3}{2}} \sigma_\nu}{2}, \quad \bar{\phi}_c(\delta t) = \omega_\nu \delta t = \frac{m_\nu^2 c^4}{2 E_\nu} \delta t. \quad (20)$$

The phase $\phi(\delta x)$ of the original expression became the small phase $\bar{\phi}_c(\delta t)$ of Eq. (20) at the light cone $\lambda = 0$. The next singular term is from $1/\lambda$ in $\Delta_{\pi,l}(\delta x)$, and becomes $J_{\delta(\lambda)}/\sqrt{\pi \sigma_\nu |\vec{p}_\nu|^2}$. This is much smaller than $J_{\delta(\lambda)}$ and is negligible in the present parameter region. The magnitude is inversely proportional to $|\delta t|$ and is independent of \tilde{m}^2 . This behavior is satisfied in general forms of the wave packets (see Appendix C).

The regular terms of $\Delta_{\pi,l}(\delta x)$ give finite contributions from the region $\vec{r} \approx 0$. The first term, \tilde{L}_1 , is from $I_1^{regular}$ in Eq. (19), and behaves as

$$\begin{aligned}\tilde{L}_1 &= C_1 |\delta t|^{-\frac{3}{4}} \exp \left[i(E_\nu - |\vec{p}_\nu| v_\nu) \delta t - \sigma_\nu |\vec{p}_\nu|^2 + i\tilde{m} \sqrt{2v_\nu \sigma_\nu} |\vec{p}_\nu| |\delta t| \right], \\ C_1 &= i \frac{\sigma_\nu}{4} \left(\frac{\sigma_\nu \tilde{m}}{2} \right)^{\frac{1}{2}} (4v_\nu \sigma_\nu |\vec{p}_\nu|)^{-\frac{3}{4}},\end{aligned}\tag{21}$$

in the space-like region $\lambda < 0$, and \tilde{L}_1 decreases with time as $\exp \left[-\tilde{m} b_1 \sqrt{|\delta t|} \right]$ in the time-like region $\lambda > 0$. The second term, \tilde{L}_2 , is from I_2 , and is approximately the integral of $\exp \left[-i \left(E_\pi - E_\nu - \sqrt{|\vec{q}|^2 + m_l^2} \right) \delta t \right]$ in \vec{q} in a range $1/\sqrt{\sigma_\nu}$. \tilde{L}_2 is a steeply decreasing function of $|\delta t|$.

Finally the probability is written in the form,

$$\begin{aligned}P &= N_3 \int d\vec{X}_\nu \int dt_1 dt_2 \left[\frac{\epsilon(\delta t)}{|\delta t|} e^{i\bar{\phi}_c(\delta t)} + 2D_{\tilde{m}}(p_\nu) \frac{\tilde{L}_1}{\sigma_\nu} - \frac{2i}{\pi} \left(\frac{\sigma_\nu}{\pi} \right)^{\frac{1}{2}} \tilde{L}_2 \right], \\ N_3 &= ig^2 m_\mu^2 \pi^3 \sigma_\nu (8p_\pi \cdot p_\nu / E_\nu) V^{-1},\end{aligned}\tag{22}$$

where $D_{\tilde{m}}(p_\nu)$ and \tilde{L}_2 are given in Appendix B. The first term in Eq. (22) oscillates extremely slowly with time δt and the remaining terms oscillate or decrease rapidly. The integral of the first term over the finite T slowly approaches constant.

Here we study the series $\sum_n (-2p_\pi \cdot p_\nu)^n \frac{1}{n!} \left(\frac{\partial}{\partial \tilde{m}^2} \right)^n \tilde{L}_1$, in Eq. (22). This series converges when $S_1 = \sum_n (-2p_\pi \cdot p_\nu)^n \frac{1}{n!} \left(\frac{\partial}{\partial \tilde{m}^2} \right)^n (\tilde{m}^2)^{\frac{1}{4}}$ becomes finite. This is satisfied in $2p_\pi \cdot p_\nu \leq \tilde{m}^2$, and the power series rapidly oscillates with $\sqrt{|\delta t|}$ as $S_2 = \exp \left[i\tilde{m} \sqrt{2v_\nu \sigma_\nu} |\vec{p}_\nu| |\delta t| \left(1 - \frac{p_\pi \cdot p_\nu}{\tilde{m}^2} \right) \right]$. Therefore the diffraction term exists in the region $2p_\pi \cdot p_\nu \leq \tilde{m}^2$ and vanishes in outside region.

Thus the light-cone singular term leads $\frac{1}{\delta t} e^{i\omega_\nu \delta t}$ and the regular term, $I_1^{regular}$, gives a finite contribution from the region $\vec{r} \approx 0$.

The integrations over times t_1 and t_2 of the singular term in a finite T is

$$i \int_0^T dt_1 dt_2 \frac{\epsilon(\delta t)}{|\delta t|} e^{i\omega_\nu \delta t} = T(\tilde{g}(\omega_\nu T) - \pi),\tag{23}$$

where the function $\tilde{g}(\omega_\nu T)$ in the right-hand side satisfies $\frac{\partial}{\partial T}\tilde{g}(\omega_\nu T)|_{T=0} = -\omega_\nu$ and $\tilde{g}(\omega_\nu T) = \frac{2}{\omega_\nu T}$, $\omega_\nu T \rightarrow \infty$. The last term in Eq. (23) is canceled by the integral of the short-range term $I_1^{regular}$. Since $\tilde{g}(\omega_\nu T)$ is generated by the superposed waves that have the form of the light-cone singularity, and its effect remains at a macroscopic distance of the order of $2c\hbar E_\nu/(m_\nu^2 c^4)$, we call this the **diffraction** term. The last term I_2 in Eq. (19) gives $\frac{2}{\pi}\sqrt{\frac{\sigma_\nu}{\pi}}\int dt_1 dt_2 \tilde{L}_2(\delta t) = TG_0$, where the constant G_0 is computed numerically. Owing to the rapid oscillation in $|\delta t|$, the microscopic $|\delta t|$ contributes to this integral, and consequently G_0 is constant in T .

Integrating over the neutrino's coordinate \vec{X}_ν , we obtain the total volume, which is canceled by the factor V^{-1} from the normalization of the initial pion state. The total probability is then expressed as the sum of the normal term G_0 and the diffraction term $\tilde{g}(\omega_\nu T)$:

$$P = N_4 \int \frac{d^3 p_\nu}{(2\pi)^3} \frac{p_\pi \cdot p_\nu (m_\pi^2 - 2p_\pi \cdot p_\nu)}{E_\nu} [\tilde{g}(\omega_\nu T) + G_0], \quad (24)$$

where $N_4 = 8Tg^2\sigma_\nu$ and $L = cT$ is the length of the decay region. The quantity P is the neutrino flux measured by its reaction with the physical state of the target with finite size σ_ν . The diffraction term corresponds to the energy-non-conserving term $S^{(d)}[T]$, and the normal term corresponds to the energy conserving term $S^{(n)}[T]$. At $T \rightarrow \infty$, the diffraction term vanishes and at a finite T , the probability contains the diffraction component, which is stable with respect to variation of the pion's momentum.

3 Unusual behavior of diffraction component

Next, we evaluate each term of Eq. (24). In the normal term G_0 , the energy and momentum are approximately well conserved, $p_\pi = p_l + p_\nu$, and satisfies $2p_\pi \cdot p_\nu - m_\pi^2 = m_l^2$. Thus, the branching ratio of the electron mode is about 10^{-4} of that of the muon mode [22, 23, 21, 24]. Integrating over the neutrino's angle, we find that this term is independent of σ_ν , which is consistent with the condition for the stationary state [17]. The transition rate agrees with the value obtained by the ordinary method. In the diffraction component, $\tilde{g}(\omega_\nu T)$, the energy and momentum are not conserved, $p_\pi \neq p_l + p_\nu$, and the inner product, $p_\pi \cdot p_\nu$, is not expressed with the masses of pion and charged lepton. Instead, the convergence condition requires that this term be present in the kinematical region, $|\vec{p}_\nu|(E_\pi - |\vec{p}_\pi|) \leq p_\pi \cdot p_\nu \leq \tilde{m}^2/2$. Consequently the

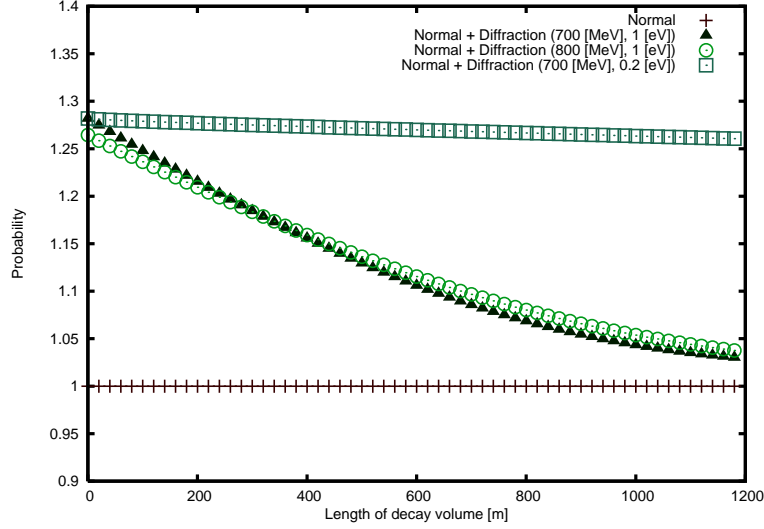


Figure 2: Total detection rate of the muon neutrino at a finite distance L . The constant (red line) shows the normal term, and the diffraction term is given on top of the normal term. The horizontal axis represents the distance in meters, and the normal term is normalized to 1.0. The neutrino mass, pion energy, and neutrino energy are 1.0 [eV/ c^2] or 0.2 [eV/ c^2], 4 [GeV], and 700 (blue triangles) or 800 (green circles) [MeV], respectively. The excess varies with the distance for $m_\nu = 1.0$ [eV/ c^2] and is almost constant for $m_\nu = 0.2$ [eV/ c^2].

electron mode is not suppressed in the diffraction component and momentum is integrated in this region.

For the muon mode, $l = \mu$, the angle of the neutrino in the diffraction component is slightly different from the angle of the neutrino in the normal mode, so it is impossible to experimentally distinguish the latter region from the former. Therefore, we add both terms, and the total probability thus obtained is presented in Fig. (2) for two neutrino masses $m_\nu = 1$ [eV/ c^2] and 0.2 [eV/ c^2], a pion energy $E_\pi = 4$ [GeV], and two neutrino energies $E_\nu = 700$ and 800 [MeV]. For the wave packet size of the neutrino, we use the size of the nucleus having a mass number A , $\sigma_\nu = A^{2/3}/m_\pi^2$. For the ^{16}O nucleus, this size takes the value $\sigma_\nu = 6.4/m_\pi^2$. From Fig. (2), we see that an excess of flux varies with the distance for distances $L < 1200$ [m] for $m_\nu = 1$ [eV/ c^2] and is almost constant for 0.2 [eV/ c^2] and that the maximal excess is approximately

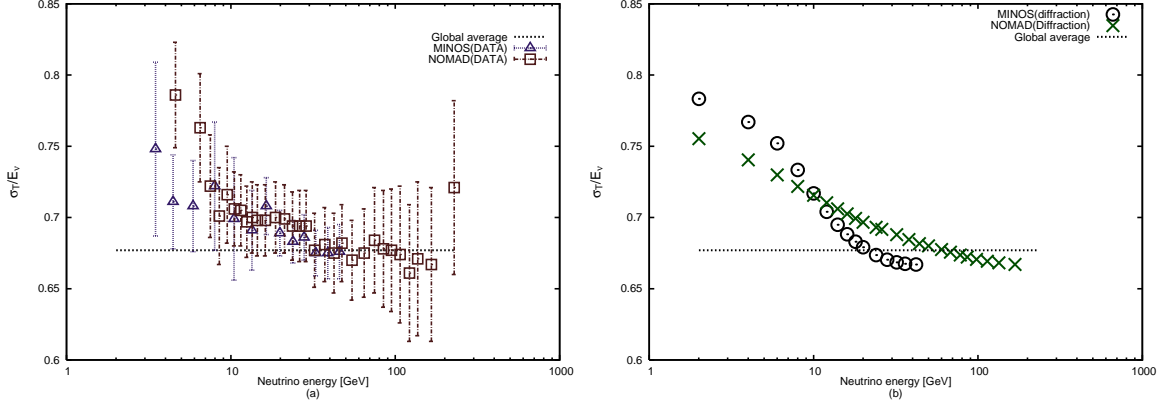


Figure 3: νN total cross section from MINOS and NOMAD collaborations (a) and total cross section of sums of normal and diffraction terms in geometries of MINOS and NOMAD collaborations (b). The horizontal axis gives the length in meters and the vertical axis gives the ratio of cross section to energy. The neutrino mass, $m_\nu = 0.2[eV/c^2]$, is used for theoretical calculation.

20% of the normal term at $L = 0$. The slope at the origin $L = 0$ is determined by ω_ν . The diffraction term varies slowly with both distance and energy. For this situation, the typical length is $L_0 [m] = 2E_\nu \hbar c / (m_\nu^2 c^4) = 400 \times E_\nu [GeV] / m_\nu^2 [eV^2/c^4]$. The neutrino's energy is measured experimentally with uncertainty ΔE_ν , which is of the order $0.1 \times E_\nu$. This uncertainty is 100 [MeV] for 1 [GeV] neutrino energy and the diffraction components of both energies are almost equivalent to those given in Fig. (2). For larger uncertainty in energy, the computation is easily done using Eq. (24).

Fig. (3) compares the total cross sections of the neutrino nucleon reactions of the NOMAD [25] and MINOS [26] collaborations. The geometry of each experiment is taken into account in the theoretical calculations. The theoretical values obtained depend slightly on energy and agree with experiments. The slight energy dependence of the total cross sections at high energy, which is difficult to understand from the standard quark-parton model, becomes understandable when considering the diffraction components.

For the electron mode, $l = e$, the finite-size correction becomes much larger than the normal term in short distance regions. This is because the normal term is suppressed by helicity suppression (which originates in energy-momentum and angular momentum conservation) [22, 23, 21, 24] whereas the diffraction term is not affected by the helicity suppression. Its magnitude

is about 0.2 of that of the muon mode from Fig. (2). Thus, the excess of rate becomes enormous. Fig. (4) shows the relative fraction of the electron mode that includes geometrical configurations of detectors. The value is 10^{-4} in the normal mode, around 0.1 at a distance of a few meters owing to the diffraction component, and around 0.01 at a distance of 10 to 100 meters. The diffraction component becomes negligible in a distance above 1000 meters. The enhancement factor is 10^3 in the former case and 10^2 in the latter case. For the LSND experiment [27], the distance is about 20 to 30 meters. The theoretical values are also plotted in Fig. (4) and agree well with the experimental values. Thus, we consider the LSND events to signal of neutrino diffraction. The diffraction in a configuration of the first high-energy neutrino experiment, TWN [28] collaboration, is also compared in the same figure and an agreement is obtained. Theoretical values depend on the neutrino mass weakly in the range $0.2 - 1.0$ [eV/ c^2] in this figure. An energy spectrum at low energy region of around 100 [MeV] or less, on the other hand, is sensitive to the mass. Further experiments at shorter distance may be able to confirm the diffraction of the electron neutrino.

Thus the probability of the detection process described by $S[T]$ expressed by Eq. (24) deviates from the probability given by $S[\infty]$ and has the large finite-size correction. If the charged lepton is observed simultaneously, the detection rate of the lepton has also the large finite-size correction. This result is different from the probability to detect only charged lepton but does not contradict with these ordinary experiments because the boundary conditions are different in two cases.

The finite-size correction of the probability to detect only a charged lepton is computed with $S[T']$ which satisfies the boundary condition for the charged lepton. Here $T' = T_\mu - T_\pi$ is the time interval for observing the charged lepton and the probability is expressed by Eq. (24) with $\omega_\nu \rightarrow \omega_l = m_l^2 c^4 / (2E_l \hbar)$. Since charged leptons are heavy, $\omega_l T'$ becomes very large and $\tilde{g}(\omega_l T')$ becomes $\frac{2}{\omega_l T} \approx 0$ for macroscopic T' . Thus, the probability to detect the charged lepton does not have a finite-size correction and agrees with that of the normal term. Although the light-cone singularity forms in both cases, the diffraction component becomes relevant only when the detected particle is very light.

The probability to detect a charged lepton depends on the boundary condition of the neutrino. When a neutrino is detected at T_ν , the charged-lepton spectrum includes the diffraction component but, when the neutrino is not detected, the charged-lepton spectrum does not include the diffraction component. The latter result is standard, whereas the former is not, but may

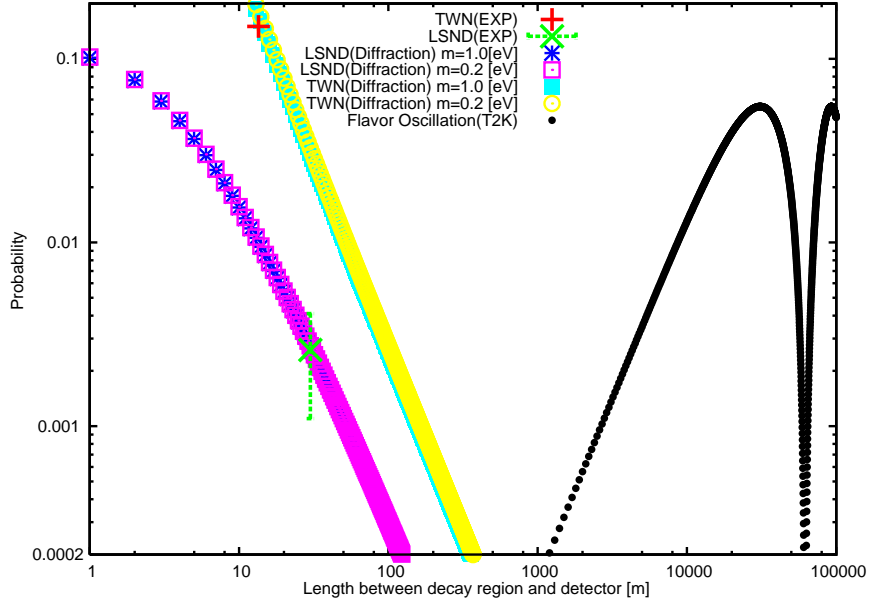


Figure 4: Relative fraction of the electron mode computed theoretically for LSND geometry and that for TWN geometry are compared with experiments of LSND and TWN [28] collaborations. TWN (EXP) and LSND (EXP) give the experimental values and TWN (Diffraction) gives the results calculated with the parameters $m_\nu = 0.2$ and 1.0 [eV/c²], $E_\nu = 250$ [MeV], and $P_\pi = 2$ [GeV/c]. LSND (Diffraction) is calculated with $m_\nu = 0.2$ and 1.0 [eV/c²], $E_\nu = 60$ [MeV], and $P_\pi = 300$ [MeV/c]. Flavor oscillation (T2K) shows the results for $\sin^2 \theta_{13} = 0.11$, $\delta m_{23}^2 = 2.4 \times 10^{-3}$ [eV²/c⁴], and $E_\nu = 60$ [MeV].

be verified experimentally.

For three neutrinos of mass m_{ν_i} and a mixing matrix $U_{i,\alpha}$, the diffraction term for a neutrino of flavor α is expressed as $\sum_i \tilde{g}(\omega_{\nu_i} T) |U_{i,\alpha}|^2$, whereas the normal term is expressed as $|\sum_i U_{i,\mu} D(i) U_{i,\alpha}^\dagger|^2$ where i is the mass eigenstate, α is the flavor eigenstate, and $D(i)$ is the free wave of m_{ν_i} . Thus, the diffraction term depends on the square of the average mass, \bar{m}_ν^2 , but the normal term depends on the mass-squared difference δm_ν^2 . At $L \rightarrow \infty$, the diffraction term disappears and the normal term that includes the flavor oscillation remains. Fig. (4) shows that the probability to detect the neutrino in the region above 1000 meters is described by the flavor oscillation and that below a few hundred meters is described by the neutrino diffraction.

We now compare neutrino diffraction with the diffraction of light passing through a hole. The former is that of a quantum mechanical wave and appears in the probability distribution. The diffraction pattern forms in a direction parallel to the momentum and with a phase difference $\omega_\nu \delta t$ of the non-stationary wave. Its size is determined by ω_ν , which is extremely small and stable with respect to variations in the energy and other parameters. Thus, without fine tuning the initial energy, the diffraction is easily observed, whereas the latter is that of a classical wave and appears in the intensity. The diffraction forms in a direction perpendicular to the momentum and with a phase difference $\omega_\gamma^{dB} \delta t$ where $\omega_\gamma^{dB} = cp/\hbar$ for the stationary wave. Its shape is determined by ω_γ^{dB} , which is large and varies when the parameters are changed. Thus, the initial energy must be fine tuned to observe the diffraction of light.

4 Summary and future prospects

We found that the wave function of the pion and decay products at a finite t has a kinetic energy of continuous spectrum and possesses wave natures which are very different from that at $t = \infty$. The rate to detect the neutrino at a finite-time interval T is uniquely computed with the wave packet and reveals a large finite-size correction. Because the wave nature causes the correction, that has unusual properties such as the non-conservation of the kinetic energy, the non-suppression of the electron mode, and others. Applying $S[T]$ that satisfies the boundary condition of a finite T , we obtained the results that can be compared with experiments. Pion and muon waves accumulating at the speed of light form the light-cone singularity, which is real and extended in the large-distance region, thus strongly influence the neutrino. The probability to detect a neutrino produced along the light-cone singularity is subject to the finite-size correction of the diffraction phenomenon. The diffraction pattern is determined by the difference of angular velocities, $\omega_\nu = \omega_\nu^E - \omega_\nu^{dB}$, where $\omega_\nu^E = E_\nu/\hbar$ and $\omega_\nu^{dB} = c|\vec{p}_\nu|/\hbar$. The quantity ω_ν takes the extremely small value $m_\nu^2 c^4/(2E_\nu \hbar)$ because of the unique neutrino features [6, 7, 8]. Consequently, the diffraction term becomes finite in the macroscopic spatial region of $r \leq \frac{2\pi E_\nu \hbar c}{m_\nu^2 c^4}$ and affects experiments in the mass-dependent manner at near-detector regions. Excess in neutrino flux in this region observed by K2K [29], MiniBooNE [30], and MINOS [31] may be connected with the diffraction component. In addition, the excess in electron

neutrinos known as the LSND anomaly may be attributed to the diffraction component instead of to flavor oscillation, which would resolve the controversy between LSND with others [32]. We compared all previous neutrino experiments and found that the new contribution in near-detector regions derived from the neutrino diffraction is surprisingly consistent with them. It would be important to confirm the neutrino diffraction with precision experiments and to find the absolute neutrino mass. If the mass satisfies $\bar{m}_\nu^2 \approx \delta m_\nu^2$ instead of $\bar{m}_\nu^2 \gg \delta m_\nu^2$, the neutrino fluxes would show more complicated behavior.

We described a new quantum phenomenon for neutrinos specific to particles of extremely small mass, and discussed the physical quantity determined by the absolute neutrino mass. In studying the quantum effects, we took the lowest order term in G_F , and ignored higher-order effects such as the pion's life-time, effects of electroweak gauge theory, and mean-free-path. The effects due to a propagator of W^\pm and higher order corrections of the renormalized theory do not modify the amplitude in the lowest order in W 's mass, M_W , and the light-cone singularity, hence our results of the paper are kept intact by them. They will be included along with other large-scale physical phenomena of low-energy neutrinos in subsequent presentations.

Acknowledgments

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A Appendix A. The finite-size correction to Fermi's Golden rule

In computing a scattering cross section and decay rate with Fermi's Golden rules, the following formula for a large T for a smooth function $g(\omega)$,

$$\int d\omega g(\omega) \left(\frac{\sin(\omega T/2)}{\omega} \right)^2 = T \int dx g(x/T) \left(\frac{\sin(x/2)}{x} \right)^2 = 2\pi T g(0), \quad (25)$$

is used [33, 34]. A $1/T$ correction to this formula is difficult to obtain with this expression, because an expansion, $g(x/T) = g(0) + \sum_l g^{(l)}(0)/l!(x/T)^l$, leads the integral over x for the coefficient of $1/T^l$ diverges for even number l .

In the computation of the finite-size correction with $S[T]$, the integral is regularized according to the boundary conditions of experiments expressed by the wave packet. Eq. (14) is such amplitude that satisfies the boundary conditions for detection of the neutrino at a finite-time interval T . Hence the finite-size correction is computed in a unique manner. The probability depends on the angular velocity ω at around a root of $\omega = 0$, where $\omega = E_l + E_\nu - E_\pi - \vec{v}_\nu \cdot (\vec{p}_l + \vec{p}_\nu - \vec{p}_\pi)$ in Eq. (15), of the text. Due to the velocity dependent term in ω , there exist roots of $\delta\vec{p} \neq 0$ and $\delta E \neq 0$.

Fig. (4) shows a solution of $\delta\vec{p} \neq 0$. A behavior of the momentum dependent amplitude Eq. (15) and that of the angular velocity ω in the parallel direction to the neutrino are shown in (a) and (b), where the angle between the lepton l and neutrino vanishes, $\theta_{l,\nu} = 0$. The angular velocity vanishes and the amplitude shows the clear peak at $E/\hbar = 128$ corresponding to the second solution, while the normal solution is not on this axis but has the energy $E/\hbar = 155$.

The roots for \vec{p}_l in the case $\delta\vec{p} = 0$ constitute an ellipse that is almost symmetric of a finite size. The angular velocity ω varies in the normal direction to the ellipse with a distance s linearly with a slope of the order of p_π or m_π . The slope is not small and the integral on the muon momentum along the normal direction converges rapidly owing to the coefficient T and the formula $\left(\frac{\sin(\omega T/2)}{\omega} \right)^2 = 2\pi T \delta(\omega)$ is valid.

The roots for $\delta\vec{p} \neq 0$ constitute also an ellipse, but the size is large and the shape is asymmetric. The major axis is of the order E_ν^2/m_ν^2 and the minor axis is of the order E_ν/m_ν . The derivative of the angular velocity ω

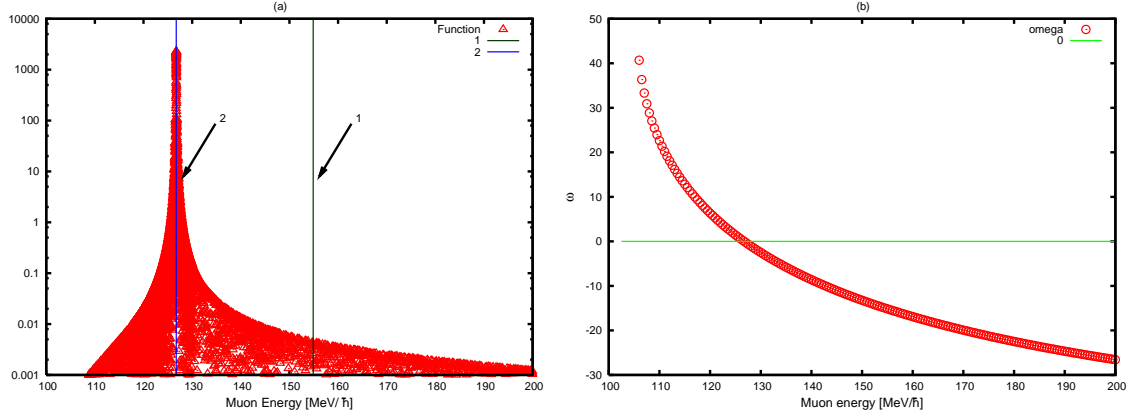


Figure 5: (a) Muon momentum dependence of the amplitude Eq. (15) and (b) the angular velocity ω of the text along the axis $\theta_{\mu,\nu} = 0$ are given. The horizontal axis shows the energy of the muon and the vertical axis show the magnitude of the amplitude in (a) and that of the ω in (b). Parameters are $\hbar = 1$, $\sigma_\nu = 1/m_\pi$, $cT=100$ [m], $E_\nu = 30$ [MeV], $E_\pi = 200$ [MeV].

along the normal direction is computed similarly and becomes as small as

$$\frac{\partial \omega}{\partial s} = \frac{m_\nu^2}{E_\nu}. \quad (26)$$

Because this slope is so small that the integral in s does not converge fast. The integral at the finite T deviates from that of the infinite T and the deviation is determined by $\frac{m_\nu^2}{E_\nu}T$, which leads the finite size correction. In the time T of $\frac{m_\nu^2}{E_\nu}T \leq 1$, the deviation is sizable.

B Appendix B. Light cone singularity.

Innumerable states at the ultra-violet energy region in a relativistic invariant system lead the correlation function $\Delta_{\pi,l}(\delta x)$ to be expressed by the four dimensional integrals of the variable $q = p_l - p_\pi$, which is conjugate to δx , over the regions $0 \leq q^0$ and $-p_\pi^0 \leq q^0 \leq 0$.

The integral over $0 \leq q^0$ is expressed as,

$$\left[m_\pi^2 p_\nu \cdot \left(p_\pi + i \frac{\partial}{\partial \delta x} \right) - 2i(p_\pi \cdot p_\nu) p_\pi \cdot \left(\frac{\partial}{\partial \delta x} \right) \right] \tilde{I}_1, \quad (27)$$

where

$$\tilde{I}_1 = \int d^4q \frac{\theta(q^0)}{4\pi^4} \text{Im} \left[\frac{1}{q^2 + 2p_\pi \cdot q + \tilde{m}^2 - i\epsilon} \right] e^{iq \cdot \delta x},$$

and $\tilde{m}^2 = m_\pi^2 - m_l^2$. By expanding the integrand of \tilde{I}_1 in $p_\pi \cdot q$, we have the expression with the light-cone singularity [35, 36], $\delta(\lambda)$, and less singular and regular terms that are described with Bessel functions,

$$\begin{aligned} \tilde{I}_1 &= 2i \left[\frac{\epsilon(\delta t)}{4\pi} \delta(\lambda) + \tilde{I}_1^{regular} \right], \quad 2i\tilde{I}_1^{regular} = D_{\tilde{m}} \left(-i \frac{\partial}{\partial \delta x} \right) f_{short}, \\ f_{short} &= -\frac{i\tilde{m}^2}{8\pi\xi} \theta(-\lambda) \{N_1(\xi) - i\epsilon(\delta t)J_1(\xi)\} - \frac{i\tilde{m}^2}{4\pi^2\xi} \theta(\lambda) K_1(\xi), \\ D_{\tilde{m}} \left(-i \frac{\partial}{\partial \delta x} \right) &= \sum_l \left(\frac{1}{l!} \right) \left(-2ip_\pi \cdot \left(\frac{\partial}{\partial \delta x} \right) \frac{\partial}{\partial \tilde{m}^2} \right)^l, \quad \xi = \tilde{m}\sqrt{\lambda}, \end{aligned} \quad (28)$$

where N_1 , J_1 , and K_1 are Bessel functions.

The integral over $-p_\pi^0 \leq q^0 \leq 0$, I_2 , has no singularity and is computed numerically.

C Appendix C. Theorem: general wave packets

In general wave packets, the leading term $J_{\delta(\lambda)}$ has the phase and the magnitude of the form Eq. (20). The phase is given as a sum of $\bar{\phi}_c(\delta t)$ and a small correction that becomes $O(1/E_\nu)$ in general and $O(1/E_\nu^2)$ in time inversion invariant wave packets.

(Proof.)

For a general wave packet, the Gaussian function is replaced with the wave function $\psi(\vec{x} - \vec{v}t)$

$$\begin{aligned} \psi(\vec{x} - \vec{v}t) &= \int dk_l d\vec{k}_T e^{ik_l(x_l - v_\nu t) + i\vec{k}_T \cdot \vec{x}_T + iC_{ij} k_T^i k_T^j t} \psi_k(k_l, \vec{k}_T), \\ C_{ij} &= \delta_{ij}/2E, \end{aligned} \quad (29)$$

where the last term in the exponent is from an expansion of $E(\vec{p} + \vec{k})$ and makes the wave packet spread with time and $\psi_k(k_l, \vec{k}_T)$ is a wave function

in the momentum. Since the coefficient C_{ij} in the longitudinal direction is negligible for the neutrino, it is neglected. The probability integrated in center coordinate \vec{X} is expressed by the square of the modules of $\psi(k_l, \vec{k}_T)$, $\tilde{\psi}(r_l - v_\nu \delta t, \vec{r}_T)$, in the form

$$\tilde{\psi}(r_l - v_\nu \delta t, \vec{r}_T) = \int dk_l d\vec{k}_T e^{ik_l(r_l - v_\nu \delta t)} e^{i\vec{k}_T \cdot \vec{r}_T + i(\vec{k}_T^2/2E)\delta t} |\psi_k(k_l, \vec{k}_T)|^2. \quad (30)$$

Then the light-cone singularity leads the slowly varying probability

$$\begin{aligned} J_{\delta(\lambda)} &= \int d\vec{r} e^{i\phi(\delta x)} \tilde{\psi}(\vec{r} - \vec{v}_\nu \delta t) \frac{\epsilon(\delta t)}{4\pi} \delta(\lambda) \\ &= \pi e^{-\frac{1}{2}} w_0 (1 + \gamma) \frac{\epsilon(\delta t)}{2\delta t} e^{i\bar{\phi}_c(\delta t)(1+\delta)}, \end{aligned} \quad (31)$$

at a large $|\delta t|$ region, where the constants are

$$\begin{aligned} w_0 &= \int dk_l |\psi_k(k_l, 0)|^2, \quad \delta = \frac{d_1}{E} + \frac{d_2}{2E^2}, \quad \gamma = \frac{d_1}{2E} + \frac{d_2}{2!} \left(\frac{1}{2E} \right)^2 - (1 - v_\nu)^2 \delta t^2, \\ d_1 &= w_0^{-1} \int dk_l k_l |\psi_k(k_l, 0)|^2, \quad d_2 = w_0^{-1} \int dk_l k_l^2 |\psi_k(k_l, 0)|^2. \end{aligned} \quad (32)$$

In high energy limit $E \rightarrow \infty$, Eq. (31) coincides with that of the Gaussian wave packet. In a wave packet of time reversal invariance, $|w(k_l, 0)|^2$ is an even function of k_l , hence $d_1 = 0$. The correction terms become negligible for $E_\nu \geq 200 - 300$ [MeV].

(Q.E.D.)

Thus the theorem is proved. This theorem demonstrates that the probability to detect the neutrino measured with the uncertainties of $\delta E \approx |\delta \vec{p}|$ by its collisions are modified with the diffraction component.